Script

Mathematics, English for Sek I and Sek II

# Mathematica - The Principles of Math

18. A World realized through Imagination, the Volume of a Circular Cone

10:17 minutes

**00:23** (caption) Sometimes, it’s used to amplify sound, …

**00:31** …or to have a stable shape that won’t fall down.

**00:38** …or to just prevent something from rolling away.

**00:42** And sometimes this shape simply happens.

**00:51** There are many reasons for the existence of the cone and pyramid.

**00:55** But how do we determine their volumes?

**01:11** Typically, the volume of a three-dimensional figure can be determined by how many unit volumes can fill it.

**01:35** But it doesn’t seem like this would work for calculating the volume of a cone or a pyramid. Isn’t there a better idea?

**01:47** The area of a triangle is half that of a rectangle with a base and height of the same length.

**01:47** (caption)
area of a triangle = base * height *1/2
area of a rectangle

**01:57** Then, do you suppose there’s a relationship between a pyramid’s volume and that of a rectangular prism of the same base and height?

**02:05** Let’s find out through a simple experiment. Let’s fill our pyramid with water and pour it into this cube that has the same base and height. After doing this three times, the cube is full of water.

**02:31** (caption) volume of pyramid *3 = volume of cube with same base and height

**02:36** But if we look more closely, we see that this experiment has some margin of error.

(caption)
volume of pyramid *3 ≒volume of cube with same base and height

**02:42** Why don’t we try to find a more precise method?

**02:51** Internal diagonal lines in a cube all meet at one point. Drawing lines from this point to the six corners of the cube, we create six congruent pyramids.

**03:04** The volume of each pyramid is exactly one-sixth the volume of the cube.

**03:16** If you draw a rectangular pyramid with the same base area and the same height as one of the pyramids, …

**03:22** …then the volume of the rectangular prism is half of the cube. That means the volume of the pyramid is one-third that of the rectangular prism.

**03:13** (caption)
volume of the pyramid = 1/6 * volume of the cube
volume of the cube = 2* volume of the rectangular prism
volume of the pyramid = 1/3 * volume of the rectangular prism

**03:40** If this is true, can we make a rectangular prism by combining three pyramids?

**03:46** It's not always the case, but if you use a figure like this one, it is sometimes possible.
How about keeping this in mind and try making one?

**04:00** If you can make three three-dimensional figures using this layout, you get three pyramids with a square base and height that is the same as one side of the square.

**04:11** And if you pile them on top of each other, you amazingly get a cube.

**04:23** Naturally, the volume of the pyramid…
… equals one-third that of the cube.

**04:26** (caption) volume of pyramid V = 1/3 * volume of cube

**04:39** Now we're starting to wonder if the 1-to-3 ratio of volume can be applied to a circular cone inside a cylinder, a triangular prism inside a triangular prism, or a hexagonal pyramid inside a hexagonal prism.

**04:46** (caption) volume ratio = 1:3 ?

**04:58** (caption)
Francesco Bonaventura Cavalieri (1598 – 1647)
Italian mathematician

**04:55** This is seventeenth century Italian mathematician Bonaventura Cavalieri. He's here to satisfy our curiosity.

**05:06** Let's make a pile of coins that are the same size. By doing this, we're creating a cylinder.

**05:13** If you nudge some part of this cylinder a little bit, the shape of the cylinder is slightly altered.

**05:19** While the position of the coins changes, their size and number do not, so the volume of the figure remains constant.

**05:31** Cavalieri applied this principle to geometric figures.

**05:37** If you keep taking a cross-section of two different three-dimensional figures, and you compare the area of the slices, you may find they always maintain a constant ratio. You can do this indefinitely, and if the ratio of the slices always remains constant, then we can say the ratio of their volumes equals the ratio of the areas of the slices.

**05:45** (caption)
area: π

**06:02** ratio of the volumes
V:V′=1:1

**06:08** The same principle can be applied to flat figures.
If you take slices of two different two-dimensional figures and the ratio of those segment slices always remains constant, then the ratio of the areas and the volumes of the two figures is the same.

**06:18** (caption)
ratio of the area
S:S′=1:1

**06:24 T**he principle behind this is called the Method of Indivisibles, or Cavalieri's Principle. It considers a three-dimensional figure to be a collection of an infinite number of two-dimensional flat figures, which themselves are an infinite collection of one-dimensional line segments.

**06:32** (caption)
Method of Indivisibles (Cavalieri's Principle)
a method in which a diagram is regarded as an infinite collection of figures of one less dimension
(e.g., three-dimensional figures are made up of an infinite collection of two-dimensional figures)

**06:41** Cavalieri applied this principle in order to find out the volumes and surface areas of three-dimensional figures. It allowed mathematicians to calculate a wide variety of figures whose surface areas and volumes were thought to be impossible due to their complicated shapes.

**06:57** Cavalieri's Principle provided the crucial key to solving the pyramid volume question.

**07:03** Going back to our earlier work, we prepared a cube and a square-based pyramid, as well as a triangular pyramid and a triangular prism whose base area and height are the same.

**07:18** Here, since the cube and the triangular prism share the same base area and height, their volumes are equal.

**07:22** (caption)
volume of the cube = S×h = S′×h = volume of the triangular prism

**07:32** What we're really concerned with are the square-based pyramid (S) and the triangular pyramid (S').

**07:39** If we take a cross-section slice of each, …

**07:42** …from the base to the uppermost point, each is a collection of similar figures.

**07:46** All the slices of the square-based pyramid S have the same sized angles and the lengths of their sides are all in a constant ratio. The same is true of the triangular pyramid S'. (READ: s-prime)

**07:57** Remember, the base area of S and S' are the same. So the area of each figure's slices gets smaller at the same rate as we go up.

**08:06** So according to Cavalieri's Principle, the volume of the square-based pyramid and the triangular pyramid are the same.

**08:06** (caption)
volume of square-based pyramid = volume of triangular pyramid

**08:12** (caption)
volume of square-based pyramid = volume of triangular pyramid
volume of cube = volume of the triangular prism
volume of square-based pyramid = 1/3 * volume of cube
volume of triangular pyramid = 1/3 * volume of cube
volume of triangular pyramid = 1/3 volume of triangular prism

**08:13** Since the volume ratio of the square-based pyramid to the volume of the cube is 1 to 3, the volume ratio of the triangular pyramid to the volume of the triangular prism is also 1 to 3.

**08:30** This is true not just of triangular pyramids, but pyramids with any type of base and circular cones, as long as the base area and the height are the same.

**08:39** Now we can say with confidence that the volume of a pyramid equals 1/3 the volume of a prism with the same base area and the same height.

**08:38** (caption)
volume of pyramid = 1/3* volume of rectangular prism

**08:51** (caption)
Archimedes (287 BC – 212 BC)
ancient Greek mathematician

**08:50** The ancient Greek mathematician Archimedes calculated the circumference of a circle through inscribed polygons and circumscribed polygons. He also calculated the surface of a sphere through an infinite number of lines, and the volume of a sphere as the sum of an infinite number of cylinders.

**09:15** (caption)
Liu Hui (3rd Century BC)

Mathematician in Cao Wei state of China

**09:15** The ancient Chinese mathematician Lie Hiu commented in The Nine Chapters on the Mathematical Arts that he calculated the volume of a pyramid using the same type of method.

**09:23** (caption)
Zu Chongzhi (429 AD – 500 AD)
mathematician during China's Liu Song Dynasty

**09:25** Mathematician Zu Chongzhi of the Liu Song Dynasty of ancient China proved the volume of a sphere just as Cavalieri did.

**09:35** (caption)
Method of Indivisibles (Cavalieri's Principle)
a method in which a diagram is regarded as an infinite collection of figures of one less dimension (e.g., three-dimensional figures are made up of an infinite collection of two-dimensional figures)

**09:44** (caption) As we reconstruct the invisible inner workings of the human body through computerized tomography (CT scan), the world of shapes once thought indivisible can now be divided through human thought and innovation.

**09:44** This opens up a whole new world of mathematics.